a small change in current produces a large change of the same sign in the wave-length. This property could be applied, for example, to give modulation of wave-length, such as is sought in wireless telephony, or to form the basis of a new method of amplification. In this case the electrical stimulus to be amplified would be introduced into the filament circuit by means of a transformer. The consequent change in wave-length would be the index showing the receipt of the stimulus.

Equal Parallel Cylindrical Conductors in Electrical Problems.

By F. J. W. WHIPPLE, M.A.

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In a recent paper* Dr. Alexander Russell has called attention to the practical importance of the electrostatic problem of determining the distribution of charge and of potential when parallel cylindrical conductors are electrified. Russell points out that the same analysis will serve for the calculation of the current density and the magnetic flux when currents of high frequency pass along the parallel conductors. The solution he gives for the case of equal and opposite charges (or currents) in the two conductors is exact, but in the general case a more elaborate investigation is necessary.

In the present paper the problem is solved by the method of conjugate functions. For the most part, the results are given in forms convenient for numerical calculation. The investigation is confined to the case in which the two cylinders are of the same size; for this case ordinary Jacobian elliptic functions suffice though in general theta-functions are required and the algebra is rather heavy. It is hardly to be supposed that a problem of such a simple character in a favourite field of investigation has not been solved before. It is desirable, however, to have the solution in an accessible form.

2. *Notation.*—The problem to be considered is virtually in two dimensions, the cylinders being represented by circular sections.

Let a be the radius of either circle, $2c$ the distance between the centres, $2f$ the distance between the limiting points Ω and Ω'.

Take the origin midway between the centres and the axis of X along the line joining them. The position of a point P may be defined by curvilinear co-ordinates u and v such that the angle $\Omega P \Omega' = u$, and $\Omega' P/\Omega P = e^{v}$, which relations are equivalent to the equation

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* "Electrical Theorems in connection with Parallel Cylindrical Conductors," 'Phys. Soc. Proc.,' vol. 31, p. 111 (1919).

$$x + \iota y = \iota f \cot(u + \iota v)/2, \tag{1}$$

or to

$$x = f\frac{\sinh v}{\cosh v - \cos u} \tag{2}$$

with

$$y = f\frac{\sin u}{\cosh v - \cos u}. \tag{3}$$

The two conducting circles are given by

$$v = \pm\alpha$$

where α satisfies the equations

$$a = f \operatorname{cosech} \alpha \tag{4}$$

and

$$c = f \coth \alpha. \tag{5}$$

3. *Equal Cylinders with Equal Charges.*—The potential function ϕ and the conjugate function ψ suggested for consideration are determined by the equation

$$\phi + \iota\psi = 4\mathrm{Q}\log\{m \operatorname{sn}[(u+\iota v)\mathrm{K}/\pi]\}, \tag{6}$$

in which Q is the charge per unit length of either cylinder, K is the real quarter-period of the elliptic function, and m is an arbitrary constant. The form of the function on the right is such that ϕ is a periodic function of u with period 2π. The condition that the potential ϕ may be constant over the surfaces $v = \pm\alpha$ is

$$\mathrm{K}'/\mathrm{K} = 2\alpha/\pi, \tag{7}$$

where $2\iota\mathrm{K}'$ is the imaginary period of the function sn.

To prove this we have

$$\operatorname{sn}(z + \iota\mathrm{K}') = \frac{1}{k \operatorname{sn} z},$$

and therefore

$$\operatorname{sn}^2\left(\frac{u\mathrm{K}}{\pi} + \frac{\iota\mathrm{K}'}{2}\right) = \frac{1}{k}\,\frac{\operatorname{sn}\left(\dfrac{u\mathrm{K}}{\pi} + \dfrac{\iota\mathrm{K}'}{2}\right)}{\operatorname{sn}\left(\dfrac{u\mathrm{K}}{\pi} - \dfrac{\iota\mathrm{K}'}{2}\right)},$$

so that, if (7) holds good, (6) gives when $v = \alpha$

$$\phi = 2\mathrm{Q}\log(m^2/k). \tag{8}$$

The function ϕ has no poles in the finite part of the plane outside the cylinders, and as it has the same constant value over the two cylinders it represents the potential when they carry equal charges.

It is of interest to notice that $e^{-2\alpha} = q$, where q, defined as $e^{-\pi \mathrm{K}'/\mathrm{K}}$, is the quantity usually denoted by that letter in the theory of elliptic functions.

4. *Potential at a Great Distance* (*charges equal*).—The potential at a great distance from the conductors is determined by the asymptotic equation

$$\phi \sim 4\mathrm{Q}\left[\log \frac{2m\mathrm{K}f}{\pi} - \log r\right], \tag{9}$$

where r is the radius vector from the origin.

If the zero of potential is on $v = \alpha$, it follows from (8) that $m = k^{1/2}$.

On this convention the potential at a great distance is given by

$$\phi \sim 4\mathrm{Q}\left[\log \frac{2k^{\frac{1}{2}}\mathrm{K}f}{\pi} - \log r\right], \tag{10}$$

and is therefore asymptotic to that due to a total charge 2Q per unit length distributed over a cylinder of radius R, where

$$\mathrm{R} = 2k^{\frac{1}{2}}\mathrm{K}f/\pi. \tag{11}$$

This equivalent radius is given in Table II.

Table I.—Dimensions.

$\sin^{-1} k$.	α.	$c/a = \cosh \alpha$.	$f/a = \sinh \alpha$.	$c/f = \coth \alpha$.
0	∞	∞	∞	1
10	3·129	11·452	11·408	1·004
20	2·429	5·715	5·627	1·016
30	2·009	3·797	3·663	1·037
40	1·705	2·843	2·661	1·068
50	1·447	2·242	2·006	1·117
60	1·228	1·853	1·560	1·188
70	1·016	1·562	1·200	1·302
80	0·789	1·327	0·873	1·521
89	0·454	1·105	0·470	2·350
90	0	1	0	∞

Table II.—Potential and Force.

$\sin^{-1} k$.	γ.	$\gamma + \alpha$.	$\gamma - \alpha$.	R/a.	A.	A a/f.	A c/f.
0	∞	∞	0	∞	1	0	1
10	3·133	6·262	0·004	4·79	0·992	0·087	0·996
20	2·423	4·872	0·014	3·39	0·970	0·172	0·985
30	2·044	4·053	0·035	2·78	0·932	0·254	0·966
40	1·769	3·474	0·064	2·42	0·879	0·331	0·940
50	1·544	2·991	0·097	2·16	0·814	0·406	0·909
60	1·380	2·608	0·152	1·99	0·733	0·470	0·871
70	1·236	2·252	0·220	1·85	0·630	0·525	0·821
80	1·106	1·895	0·317	1·74	0·510	0·584	0·776
89	0·972	1·426	0·518	1·63	0·301	0·642	0·709
90	0·903	0·903	0·903	1·57	0	0·667	0·667

5. *Distribution of Lines of Force, etc.*—To determine the distribution of the charge on either cylinder, a knowledge of the variation in ψ is necessary.

Equation (6) may be written

$$\phi+\iota\psi = 2\,\mathrm{Q}\log\left[(m/k)^2(\mathrm{ns}\,2w-\mathrm{ds}\,2w)(\mathrm{ns}\,2w-\mathrm{cs}\,2w)\right] \tag{12}$$

where $w \equiv (u+\iota v)\,\mathrm{K}/\pi$.

On writing $v = \alpha = \pi\mathrm{K}'/2\mathrm{K}$ it is found that at points on the cylinder

$$\phi = 2\,\mathrm{Q}\log m^2/k$$

as before, whilst

$$\psi = 2\,\mathrm{Q}\left[(2n+1)\pi-\tan^{-1}\left(\mathrm{sc}\,\frac{2u\mathrm{K}}{\pi}\right)-\tan^{-1}\left(k\,\mathrm{sd}\,\frac{2u\mathrm{K}}{\pi}\right)\right] \tag{13}$$

where n is an integer.

In passing round the cylinder u increases continually, and so does the first inverse tangent, but the second one oscillates about zero.

If we take $n = 0$, the formula gives

when $u=0$, $\psi=2\,\mathrm{Q}\pi$; when $u=\pi$, $\psi=0$; when $u=2\pi$, $\psi=-2\,\mathrm{Q}\pi$.

6. *Density of the Charge.*—To find σ, the density of the charge, Equation 12 may be differentiated.

Since $\sigma = -\dfrac{1}{4\pi}\dfrac{\partial\psi}{\partial n}\cdot\dfrac{\partial n}{\partial s}$, where s is the arc on the circumference of the cylinder and ∂n is the element of the normal, it follows that

$$\sigma = \frac{\mathrm{K}}{\pi^2}\frac{\mathrm{Q}}{f}\left(\mathrm{dn}\,\frac{2u\mathrm{K}}{\pi}+k\,\mathrm{cn}\,\frac{2u\mathrm{K}}{\pi}\right)(\cosh\alpha-\cos u). \tag{14}$$

The mean surface density $\bar{\sigma}$ is given by

$$\bar{\sigma} = \mathrm{Q}/(2\pi a). \tag{15}$$

It follows that σ_0, the density at the points of either cylinder furthest from the other, and σ_1, the density at the points which are nearest, are related to $\bar{\sigma}$ by the equations

$$\sigma_0 = \frac{2\mathrm{K}}{\pi}(1+k)\tanh\frac{\alpha}{2}\cdot\bar{\sigma} \tag{16}$$

and

$$\sigma_1 = \frac{2\mathrm{K}}{\pi}(1-k)\coth\frac{\alpha}{2}\cdot\bar{\sigma}, \tag{17}$$

respectively.

The ratio of these densities is given by

$$\sigma_0/\sigma_1 = \left\{\tanh\frac{\alpha}{2}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}^2 \tag{18}$$

where $\sin\theta = k$.

The ratios $\sigma_0/\bar{\sigma}$ and $\sigma_1/\bar{\sigma}$ are tabulated in Table III. They are graphed as functions of c/a in fig. 1.

Table III.—Concentration of Charge.

$\sin^{-1} k$.	Equal charges.		Opposite charges.		Only one cylinder charged.			
					Charged.		Uncharged.	
	$\sigma_0/\bar{\sigma}$.	$\sigma_1/\bar{\sigma}$.	$\sigma_0/\bar{\sigma}$.	$\sigma_1/\bar{\sigma}$.	$\sigma_0/\bar{\sigma}$.	$\sigma_1/\bar{\sigma}$.	$\sigma_1/\bar{\sigma}$.	$\sigma_0/\bar{\sigma}$.
0	1	1	1	1	1	1	0	0
10	1·084	0·909	0·916	1·091	1·000	1·000	−0·091	0·084
20	1·160	0·810	0·838	1·193	0·999	1·001	−0·192	0·161
30	1·229	0·703	0·763	1·309	0·996	1·006	−0·303	0·233
40	1·291	0·585	0·692	1·444	0·991	1·015	−0·430	0·300
50	1·347	0·466	0·619	1·616	0·983	1·041	−0·575	0·364
60	1·401	0·336	0·547	1·828	0·974	1·082	−0·746	0·427
70	1·449	0·205	0·468	2·135	0·958	1·170	−0·965	0·490
80	1·494	0·081	0·375	2·67	0·934	1·37	−1·30	0·560
89	1·545	0·002	0·223	4·48	0·884	2·24	−2·24	0·660
90	1·571	0	0	∞	0·785	∞	− ∞	0·785

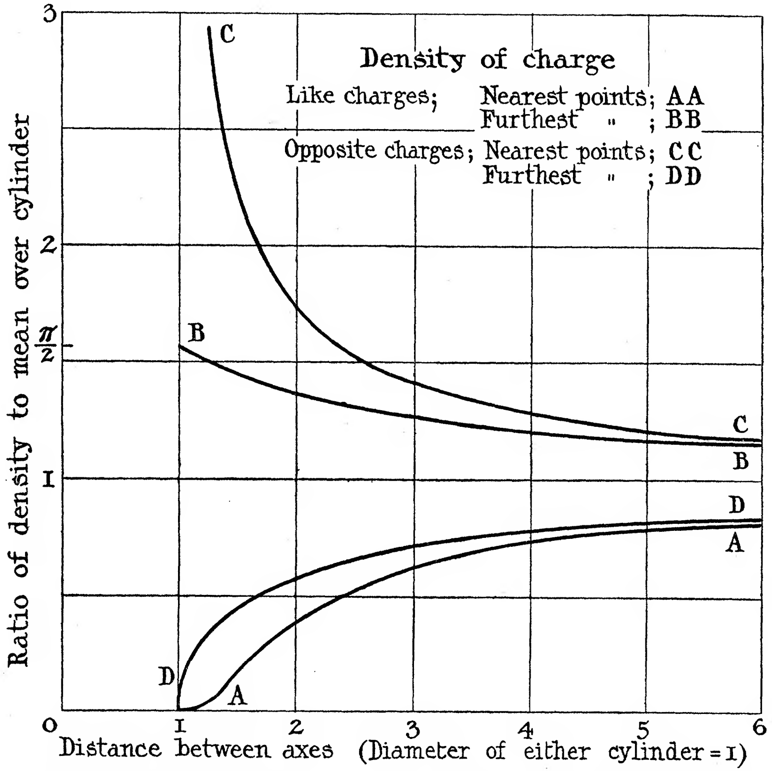

Fig. 1.—Surface densities at points on a common diameter of cylinders carrying equal charges. (Mean density over a cylinder unity.)

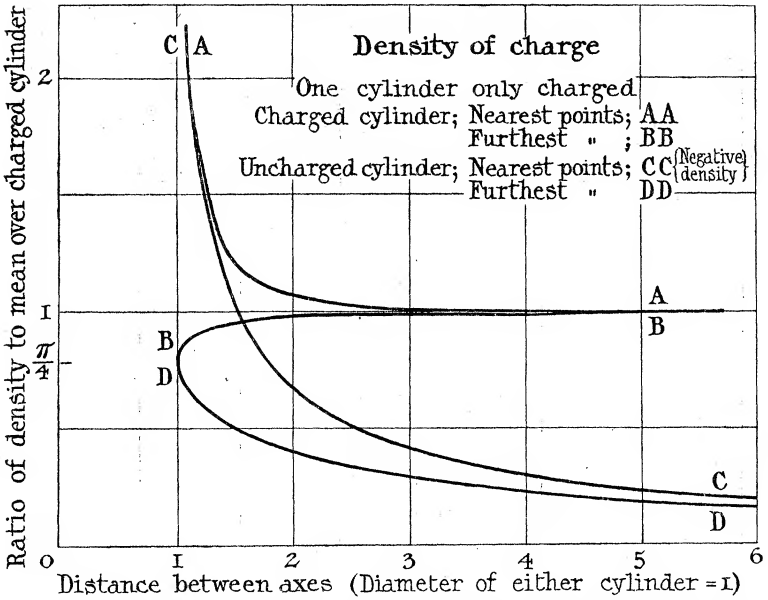

FIG. 2.—Surface densities at points on a common diameter of two cylinders, one charged, the other uncharged. (Mean density on charged cylinder unity.)

7. *Potential on the Axes.*—The way in which the potential ϕ varies along the axes is of interest:—

(a) For $u = 0$, *i.e.*, for the continuation of a common diameter of the cylinders, the equation (6) gives

$$\psi = 2Q\pi; \qquad \phi = 4Q\log\{m \,.\, \mathrm{sc}'(vK/\pi)\}, \tag{19}$$

where the dash is to indicate that the complementary modulus K′ is used.

(b) On the part of the common diameter intercepted between the cylinders $u = \pi$, and therefore

$$\psi = 0; \qquad \phi = 4Q\log\{m\,\mathrm{nd}'(vK/\pi)\}. \tag{20}$$

(c) On the axis of Y, $v = 0$, and therefore

$$\psi = 0; \qquad \phi = 4Q\log\{m\,\mathrm{sn}(uK/\pi)\}. \tag{21}$$

8. *Potential on the Orthogonal Circle.*—As an additional guide for drawing the equipotentials and lines of force, their intersections with the circle, which cuts the conductors orthogonally and symmetrically, are required. The circle is specified by $u = \pi/2$.

Over this circle ϕ and ψ are given by

$$\phi = 2Q\log\{m^2/(\mathrm{dn}'\,2vK/\pi + k'\,\mathrm{cn}'\,2vK/\pi)\}, \tag{22}$$

and $$\psi = 2Q[2n\pi - \sin^{-1}(k'\,\mathrm{sn}'\,2vK/\pi)],$$

which may be derived from (12).

9. *Repulsion between Cylinders Carrying Equal Charges.*—The repulsion per unit length is F where

$$F = 2\pi\int\sigma^2\frac{dy}{ds}\,ds \tag{23}$$

the integral being taken round the circumference.

On substitution it is found that

$$F = \frac{4K^2Q^2}{\pi^3 f}\int_0^\pi\left(\mathrm{dn}\,\frac{2uK}{\pi}+k\,\mathrm{cn}\,\frac{2uK}{\pi}\right)^2(\cosh\alpha\cos u-1)\,dn. \tag{24}$$

On integration this equation reduces to

$$F = AQ^2/f$$

where
$$A = 2\left[\coth\alpha-\left(\frac{2K}{\pi}\right)^2\left(\frac{E}{K}-\frac{k'^2}{2}\right)\right], \tag{25}$$

E being the second elliptic integral $\int_0^K \mathrm{dn}^2 z\,dz.$

The formula (25) has been used in computing the repulsion in Table II. The results are shown graphically in fig. 3.

It will be seen that for cylinders whose distance is large compared with their radii,

$$F \longrightarrow Q^2/f.$$

On the other hand when the cylinders are very close

$$F \longrightarrow 2Q^2/3a,$$

so that the repulsion between equal cylinders in contact carrying equal charges is the same as that between line charges at a distance equal to three radii, a result which can be verified by independent analysis of this important special case.*

10. *Equal Cylinders; Any Charges.*—By combining the foregoing solution for equal positive charges with the well-known one for equal and opposite charges, viz., $\phi+\iota\psi = -2\iota Q(u+\iota v)$, the distribution of potential in the general case can be found.

If the charges are Q_1 on $v=\alpha$ and Q_2 on $v=-\alpha$, and if the zero of potential is at the origin, then

$$\phi+\iota\psi = 2(Q_1+Q_2)\log\mathrm{sn}\,\{(u+\iota v)K/\pi\}-\iota(Q_1-Q_2)(u+\iota v). \tag{26}$$

The potentials of the two cylinders are given by

$$\left.\begin{aligned}\phi_1 &= (Q_1+Q_2)\log(1/k)+(Q_1-Q_2)\,\alpha\\ \phi_2 &= (Q_1+Q_2)\log(1/k)-(Q_1-Q_2)\,\alpha\end{aligned}\right\}, \tag{27}$$

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* The solution depends on the equation

$$\phi+\iota\psi = 4Q\log\tan\left(\frac{e^{-\iota\theta}a}{r}\cdot\frac{\pi}{2}\right).$$

whilst the potential at a great distance r is given by

$$\phi \sim 2(Q_1+Q_2)\log\left(\frac{2K}{\pi}\cdot\frac{f}{r}\right). \tag{28}$$

The surface-density* of the charge on the cylinder $v = \alpha$ can be computed from the formula

$$\sigma = \left[\frac{2K}{\pi}(Q_1+Q_2)\left(\text{dn}\,\frac{2uK}{\pi}+\text{cn}\,\frac{2uK}{\pi}\right)+(Q_1-Q_2)\right]\frac{\cosh\alpha-\cos u}{f}, \tag{29}$$

and the repulsion between the cylinders is determined by

$$F = \frac{A(Q_1+Q_2)^2}{4f}-\frac{(Q_1-Q_2)^2}{4f} \tag{30}$$

where A is defined by (25).

The force is negative, *i.e.*, there is an attraction when Q_1-Q_2 is large compared with Q_1+Q_2. The condition for the force to vanish is

$$\left(\frac{Q_1-Q_2}{Q_1+Q_2}\right)^2 = A. \tag{31}$$

When one of the cylinders is uncharged, so that $Q_2 = 0$, there is an attraction

$$-F = \frac{Q_1^2}{4f}[1-A]. \tag{32}$$

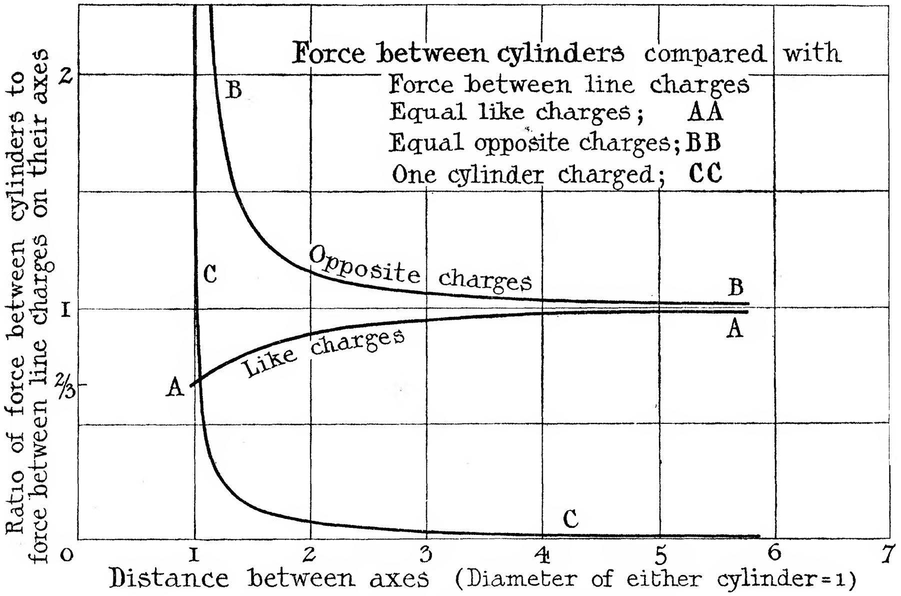

Fig. 3.—The force between two cylinders.

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* For the case $Q_2 = 0$ the surface densities at the points where $u = 0$ and $u = \pi$ are tabulated in Table III and shown graphically in fig. 2.

11. *Potential and Capacity Coefficients.*—In the ordinary treatment of potential in three-dimensional problems, it is taken as zero at a great distance from the conductors under consideration. When dealing with cylinders which are regarded for the purpose of analysis as infinite, the difference of potential between them and an object at a great distance increases as the logarithm of the distance and this distance must appear explicitly in the formulæ.

In the present problem we are virtually compelled to complete our system of two cylinders with charges Q_1 and Q_2, by supposing them to be surrounded by a very large cylinder, which may be supposed to have a radius r, and to be symmetrically placed with regard to them, and to have zero potential. It carries on its inner surface the charge $-(Q_1+Q_2)$.

The potential of the first cylinder is now given by

$$\phi_1 = 2(Q_1+Q_2)\log\left(\frac{1}{k^{\frac{1}{2}}}\frac{\pi}{2K}\frac{r}{f}\right)+(Q_1-Q_2)\alpha, \tag{33}$$

so that if

$$\left.\begin{aligned} \phi_1 &= p_{11}Q_1+p_{12}Q_2 \\ \phi_2 &= p_{12}Q_1+p_{22}Q_2 \end{aligned}\right\} \tag{34}$$

we have

$$\left.\begin{aligned} p_{11} = p_{22} &= 2\log(r/a)-(\gamma-\alpha) \\ \text{and}\qquad p_{12} &= 2\log(r/a)-(\gamma+\alpha) \end{aligned}\right\} \tag{35}$$

γ being written for

$$2\log\left(k^{\frac{1}{2}}\frac{2K}{\pi}\frac{f}{a}\right), \tag{36}$$

or in the notation of § 4 for $2\log R/a$.

Numerical values of the coefficients γ, $\gamma+\alpha$, $\gamma-\alpha$, are set out in Table II and shown graphically in fig. 4.

The charges can be written down in terms of the potentials in the form

$$\left.\begin{aligned} Q_1 &= K_{11}\phi_1+K_{12}\phi_2 \\ Q_2 &= K_{12}\phi_1+K_{12}\phi_2 \end{aligned}\right\}$$

where the capacity coefficients are given by the relations

$$\left.\begin{aligned} K_{11} = K_{22} &= \frac{2\log r/a-(\gamma-\alpha)}{4\alpha\{2\log r/a-\gamma\}} \\ K_{12} &= -\frac{2\log r/a-(\gamma+\alpha)}{4\alpha\{2\log r/a-\gamma\}} \end{aligned}\right\}. \tag{37}$$

It will be noticed that as $r \longrightarrow \infty$ the capacity coefficients remain finite ;

$$K_{11} = K_{22} \longrightarrow -K_{12} \longrightarrow 1/4\alpha.$$

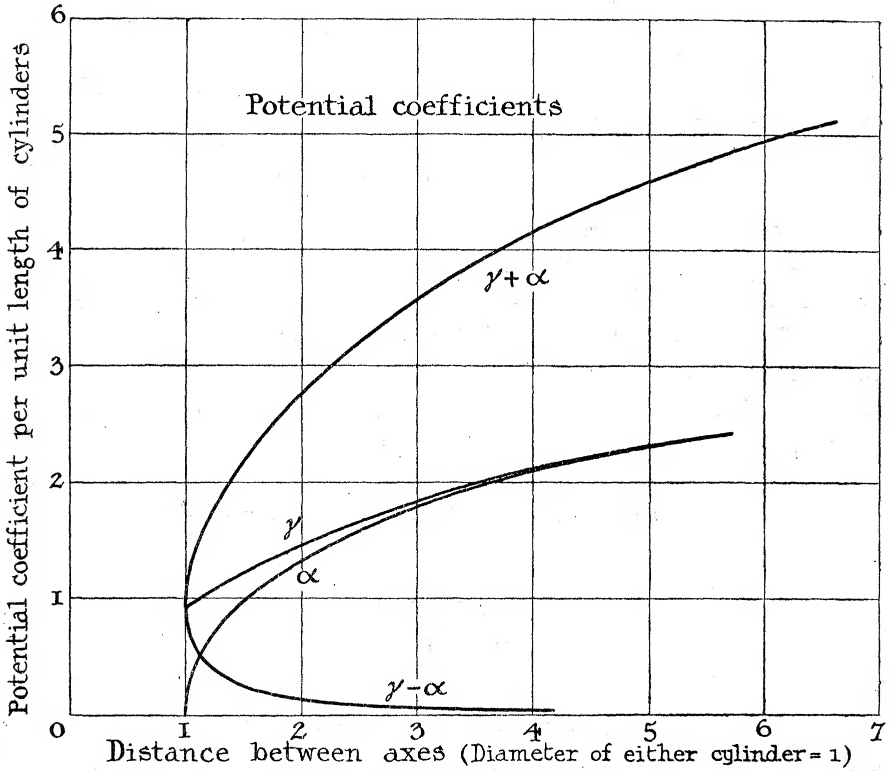

FIG. 4.—Potential coefficients.

These relations are not of direct service however: they merely indicate that, as long as $(\phi_1+\phi_2)/(\phi_1-\phi_2)$ is small compared with $\log r/a$, the total charge on the two cylinders is negligible and that

$$Q_1 = -Q_2 = (\phi_1-\phi_2)/4\alpha \text{ approximately.}$$

It is of interest to compare the potential coefficient p_{11} with that of a single isolated cylinder of radius a, which is $2\log r/a$. The difference $2\log r/a-p_{11}$ is equal to $\gamma-\alpha$ and, therefore, independent of r. This indicates that bringing another conducting cylinder to the neighbourhood of a charged cylinder changes the potential of the latter by a finite amount. Calculation shows that as long as the distance is considerable the effect is small. When the gap between the cylinders is equal to the diameter of either, the difference is only 11 per cent. of its value when the gap is infinitesimal.

12. The electrostatic energy of the system per unit length is V where

$$V = \tfrac{1}{2}(p_{11}Q_1^2+2p_{12}Q_1Q_2+p_{22}Q_2^2)$$

$$= \tfrac{1}{2}[\{2\log r/a-\gamma\}(Q_1+Q_2)^2+\alpha(Q_1-Q_2)^2]. \quad (38)$$